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Revealed preference domains from random choice ☆

Kremena Valkanova

ETH Zurich, Department of Management, Technology, and Economics, Leonhardstrasse 21, 8092 Zurich, Switzerland

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ABSTRACT

Ordinal random utility models (RUMs) are based on the presumption that fluctuating preferences drive stochastic choices. We study a novel property of RUM subclasses called exclusiveness, satisfied whenever the supports of all RUM representations of stochastic choice data, rationalizable by a RUM over preferences within a specific domain, also belong to that domain. We demonstrate that well-known preference domains such as the single-peaked, single-dipped, triple-wise value-restricted and peak-monotone are RUM-exclusive, alongside a novel domain we term peak-pit on a line. Building on existing characterization results, we show how these preference domains can be directly revealed from stochastic choice data, without the need to compute all RUM representations.

1. Introduction

Ordinal random utility models (RUMs), proposed by Block and Marschak (1960), are based on the presumption that decision-makers choose based on an unknown probability distribution over deterministic preferences. These models hold meaningful interpretations in two main contexts.[1] RUMs can capture the probability over decision-maker's diverse preference states, which might be influenced by external shocks, personal characteristics, or emotional states, resulting in stochastic choices. Alternatively, RUMs can represent the fraction of various subgroups within a society, each characterized by its deterministic preference. In this scenario, the probability distribution over alternatives is interpreted as the share with which the available options are chosen by the society.

Despite the model's simplicity and numerous applications, conducting revealed preference analysis within the described ordinal framework is challenging due to the inherent issue of underidentification when there are more than three alternatives (Barberà and Pattanaik, 1986; Fishburn, 1998; Turansick, 2022). The degree of underidentification becomes substantial with a larger set of alternatives and the multiple representations cannot be easily obtained from one another.[2] The identification problem extends in particular to recovering the support of the rationalizing distribution. There might even be RUMs with disjoint supports that generate the same set of choice probabilities (Fishburn, 1998).

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E-mail address: kvalkanova@ethz.ch.

[1] McClellon (2015) suggests a novel third interpretation according to which a RUM captures the uncertainty of an expert about the true preference of an individual or a population whose behavior she attempts to predict.

[2] See Sher et al. (2011) for precise calculations and description of the linear programming approach which can be used to compute all RUMs rationalizing a particular choice function and Footnote 6 regarding the possibility to obtain one representation from another through relabeling of the alternatives.

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The objective of this paper is to investigate whether preference domains can be unambiguously inferred from stochastic choices by verifying certain properties of the data. Thus, the type of agent's preferences can be revealed despite the multiple RUM representations and without the need to find a rationalizing RUM. For this approach to be valid, the preference domain must possess a property we call *RUM-exclusiveness*, whereby if the support of a rationalizing distribution belongs to the domain, all other observationally equivalent RUMs also belong to that same domain. Furthermore, the properties of the corresponding choice function need to be determined. In this paper, we analyze various preference domains, with a particular focus on the single-peaked domain and its generalizations, to assess their RUM-exclusiveness and the potential for these domains to be revealed from choice data.

Being able to reveal the preference domain from observed stochastic choices is useful for prediction purposes when there is a perturbation of the state probabilities. For example, consider a UX researcher who tests different layouts for an e-commerce platform, such that each layout induces a different preference relation over the available items in a consumer agent.[3] Suppose that the aggregated choice frequencies reveal the property of the agent's preferences. Can the researcher predict the choice frequencies of the consumer for a different probability distribution of the layouts? Since the set of preferences is preserved, the researcher can use the corresponding property of the stochastic choice data to learn about the new choice frequencies. Depending on the preference domain, this would make some choice experiments unnecessary, or even make it possible to predict the choice probabilities from all large sets of alternatives using binary choice data only.

Our first main result is related to the single-peaked and single-dipped preference domains (Black, 1948). RUMs are single-peaked according to some linear order if each preference in the support is such that alternatives positioned on either side of the peak are more preferred the closer their position is to the peak. This property has found numerous applications such as in models of public good provision and the Hotelling-Downs model of political competition. Single-dipped preferences possess desirable characteristics similar to those of single-peaked preferences (Barberà et al., 2012; Manjunath, 2014). The domain arises naturally when alternatives are public bads. We show that the single-peaked and single-dipped domains are RUM-exclusive, i.e. there cannot exist two observationally equivalent RUMs, with one having a single-peaked (single-dipped) support while the other does not.

Next, we consider the single-crossing domain (Mirrlees, 1971; Spence, 1974). As Apesteguia et al. (2017) prove, whenever a dataset is rationalizable by a single-crossing RUM, it is unique, which implies that the domain is not RUM-exclusive. This raises the question: is there a larger RUM-exclusive domain, which generates the same type of choice data? We demonstrate that such stochastic choices reveal that the preferences of the agent belong to a domain, which we call peak-pit on a line. A preference profile is peak-pit with respect to a linear order of the alternatives when all preferences over each triple independently are either single-peaked or single-dipped with respect to the same order.

In addition, we explore the relations between the peak-pit on a line domain and existing properties. We show that it is less general than Danilov et al. (2012)'s tiling or (local) peak-pit domain, Sen (1966)'s triple-wise value-restriction and Barberà and Moreno (2011)'s peak-monotonicity. Therefore, the peak-pit on a line domain inherits all important properties that are known to hold on these larger domains such as the existence of a Condorcet winner and a transitive majority preference, while also being able to accommodate multiple-peaked preferences. We show that local peak-pit, triple-wise value-restricted and peak-monotone preferences are also RUM-exclusive and can thus be revealed from choice frequencies.

Having established the RUM-exclusiveness of the various preference domains, we study the properties of their respective stochastic choice functions. In particular, building of the characterization results by Apesteguia et al. (2017), we show that *all* of the multiple RUMs rationalizing a stochastic choice dataset, which satisfies regularity and their strong centrality axioms, are single-peaked on a line. Similarly, if stochastic choices satisfy regularity and Apesteguia et al. (2017)'s extremality, then the preferences are revealed to be single-dipped. Since single-crossing and peak-pit on a line RUMs rationalize the same data, it satisfies regularity and centrality. Finally, we provide relaxations of the centrality axiom that correspond to the larger domains of local peak-pit, triple-wise value-restricted and peak-monotone preferences.

We conclude the introduction with a discussion of the relevant literature. Given the importance of the single-peaked domain, researchers have long been concerned with ways to verify if agents have single-peaked on a line preferences. Much of the literature on the topic tests for single-peakedness when the individual preferences in a profile are known or can be elicited and identifies different algorithms and conditions that ease this task (Ballester and Haeringer, 2011; Bartholdi and Trick, 1986; Conitzer, 2009; Doignon and Falmagne, 1994; Escoffier et al., 2008; Puppe, 2018). We contribute to this literature by identifying single-peaked preferences when the exact preference profile cannot be elicited and only aggregated choice data is observable. Moulin (1984) and Bossert and Peters (2009) characterize a family of deterministic choice functions for which the direct revealed preference relation is single-peaked. Our study shares the same objective, but within a stochastic choice context and across several generalizations of the single-peaked domain.

The primary focus of the literature on RUMs has been the problem of rationalizability of stochastic choice functions. The general characterization problem was posed by Block and Marschak (1960) and later solved in Falmagne (1978), Barberà and Pattanaik (1986), McFadden and Richter (1990), Clark (1996), and Fiorini (2004). Several more recent papers focus on the rationalizability problem for subclasses of ordinal RUMs, in particular with single-peaked or single-crossing support. While Apesteguia et al. (2017) assume that choice data on all subsets of alternatives is observable, Dridi (1994), Smeulders (2018), and Petri (2023) obtain characterization results for choice data only from binary menus.

[3] Choice predictions in response to a state probability perturbation are possible even when the exact state probabilities are not known as opposed to the example of the UX researcher. For instance, when preferences are influenced by factors like the weather, a researcher could forecast the choices of someone relocating to a different climate. In the population choice context of the RUM model, our findings can similarly predict the impact of immigration or an ageing population on choice behavior.

Another main question in the literature of RUMs is the one of identification, i.e. which types of stochastic choice function are only rationalizable by a single RUM. The problem can be circumvented if more structure is assumed on the specific types of alternatives and utility functions, which guarantee a unique RUM representation. Several examples include Gul and Pesendorfer (2006) and Lin (2020) modeling random choice over lotteries and assuming expected utility functions or weighted utility functions respectively, Lu and Saito (2018) studying choice over consumption streams and discounted utility functions, and Yang and Kopylov (2023) considering choice over bundles and quasi-linear preferences. Turansick (2022) takes a different approach to the identification problem by characterizing the stochastic choice datasets that have a unique RUM representation and RUMs with uniquely identifiable support. Our paper contributes to the literature on RUMs by focusing on the novel question of the so-called exclusiveness of a sub-class of RUMs, i.e. for a given family of stochastic choice functions, any possible RUM representation of a stochastic choice function has a support in the same preference domain.

The remainder of the paper is structured in the following way. In the next section we formalize the model and define the exclusiveness property. Section 3 contains results on the RUM-exclusiveness of various preference domains, notably the single-peaked and peak-pit on a line preferences. In Section 4 we focus on the properties of the stochastic choice functions of the exclusive RUM subclasses we previously analyzed. Section 5 concludes.

2. Modeling framework

Let X be a finite, strictly ordered set of alternatives with respect to the linear order L. A menu M is a non-empty subset of X and the set of all menus is denoted by $\mathcal{M}$.

A stochastic choice function is a mapping $\rho : X \times \mathcal{M} \to [0, 1]$ such that $\sum_{x\in M} \rho(x, M) = 1$ and $x \notin M$ implies $\rho(x, M) = 0$. The function $\rho(x, M)$ is interpreted as the probability to choose alternative x from menu M. The stochastic choice function is observable since it captures the choices of the decision maker.[4]

A preference P is defined as a complete, transitive and antisymmetric[5] binary relation over X. An alternative $x \in M$ is called a greatest element from M with respect to P if xPy, for all $y \in M \setminus \{x\}$. The set of greatest elements from a menu M with respect to P is $t(M, P)$. We denote the set of all preferences by $\mathcal{P}$. A random utility model (RUM) μ is defined as a probability distribution over $\mathcal{P}$, formally $\mu \in \Delta(\mathcal{P})$. The support of a RUM is the collection of all preferences chosen with positive probability, i.e. $\mathcal{P}_\mu = \{P \in \mathcal{P} : \mu(P) > 0\}$.

A RUM generates ρ as follows. First, a preference $P \in \mathcal{P}$ is realized with probability $\mu(P)$. Then, the greatest element $t(M, P)$ from a given menu M is chosen. A stochastic choice function that can be generated by a RUM μ is called *rationalizable* and satisfies the equality

$$\rho(x, M) = \sum_{\substack{P\in\mathcal{P}_\mu :\\ x\in t(M,P)}} \mu(P), \tag{1}$$

for all $x \in M$ and all $M \in \mathcal{M}$. In other words, the probability to choose a given element x from a menu M is given by the probability that a preference with greatest element x is realized.

Finally, we say that a RUM possesses a preference property if its support satisfies the preference property. We define below the novel exclusiveness property of RUM subclasses.

Definition 1. A preference domain is RUM-exclusive when, if a stochastic choice function is rationalizable with a RUM with the given property, then any other rationalizing RUMs that may exist also possess the same preference property.

Note that the universal preference domain is trivially RUM-exclusive. Similarly, singleton domains are also trivially RUM-exclusive. The following example adapted from Fishburn (1998) contains binary domains, which are not RUM-exclusive.

Example 1. Consider a set of alternatives $X = \{w, x, y, z\}$ and the RUMs μ and μ' with supports $\mathcal{P}_\mu = \{P_1, P_2\}$, $\mathcal{P}_{\mu'} = \{P'_1, P'_2\}$, where $xP_1yP_1wP_1z$, $yP_2xP_2zP_2w$, $xP'_1yP'_1zP'_1w$, $yP'_2xP'_2wP'_2z$, and $\mu(P_1) = \mu'(P'_1) = 0.5$. Note that the generated stochastic choice functions of the two RUMs overlap on all menus even though the corresponding supports are disjoint, for example $\rho(w, \{w, z\}) = \mu(P_1) = \rho'(w, \{w, z\}) = \mu'(P'_2) = 0.5$.[6] Thus, each of the binary domains $\mathcal{P}_\mu$ and $\mathcal{P}_{\mu'}$ are not RUM-exclusive.

[4] As mentioned earlier, RUMs have a meaningful interpretation both in individual and group decision-making contexts. Therefore, $\rho(x, M)$ can also be interpreted as the fraction of the society that chooses x from a menu M. For consistency reasons, we will use mostly the individual stochastic choice interpretation throughout the paper.

[5] Note that the assumption of strict preferences in the RUM context is not a restrictive one since every RUM with weak preferences can be represented as one with only strict preferences that generates the same stochastic choice function. However, this assumption does impact our results, which we address in Section 4.

[6] In Example 1, the two rationalizing models do not differ from each other beyond the potential relabeling of the alternatives, i.e. we can obtain μ' by relabeling z and w in μ. Such straightforward derivation of one rationalizing model from others is not possible in general, even within the class of single-peaked RUMs. Consider for example a RUM μ^* with support $\mathcal{P}_{\mu^*} = \{P_1, P_2, P_3\}$ with P_1 and P_2 defined as in Example 1 and $yP_3wP_3xP_3z$ and $\mu^*(P_1) = \mu^*(P_2)$. Observe that the RUM μ^* generates the same choice function as a RUM μ^{**} with support $\mathcal{P}_{\mu^{**}} = \{P'_1, P'_2, P_3\}$ and $\mu^*(P_1) = \mu^{**}(P'_1) = \mu^{**}(P'_2)$. The RUM μ^{**} cannot be obtained from μ^* by relabeling of alternatives.

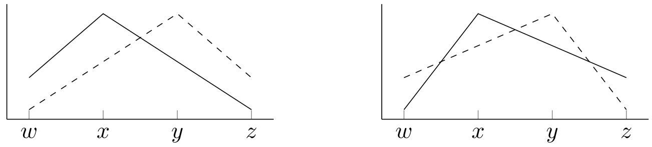

Fig. 1. Single-peaked RUMs with supports $\mathcal{P}_\mu$ (left) and $\mathcal{P}_{\mu'}$ (right).

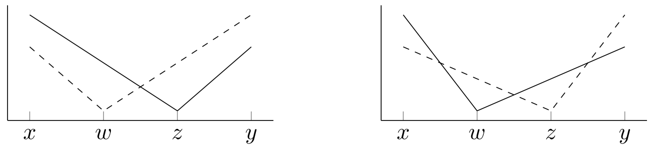

Fig. 2. Single-dipped RUMs with supports $\mathcal{P}_\mu$ (left) and $\mathcal{P}_{\mu'}$ (right).

In what follows, we examine the exclusiveness property of various preference domains, which lie between the extreme cases of the universal and singleton domains.

3. Exclusiveness of random utility submodels

Our analysis of the RUM-exclusiveness of preference domains is centered around well-known and widely used domains such as the single-peaked, single-dipped on a line and single-crossing and generalizations thereof.

3.1. Single-peaked and single-dipped domains

First, we focus on ordinal random utility models (RUMs) with single-peaked and single-dipped supports. Recall the definition of single-peakedness on a line for strict preferences.

Definition 2. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is single-peaked w.r.t. the linear order L if for every $P \in \mathcal{Q}$ and every triple $\{x, y, z\} \subseteq X$ with $z \in t(\{x, y, z\}, P)$:

$$[xLyLz \text{ or } zLyLx] \implies yPx.$$

Put differently, each preference in a single-peaked profile has a unique greatest element, or peak, and for all pairs of alternatives positioned on either side of the peak according to L the less preferred alternative in the pair is the one positioned farther away from the peak of the preference.

Single-dippedness is a preference property which is inversely related to single-peakedness. Let $b(M, P)$ denote the bottom ranked alternative according to a preference $P \in \mathcal{P}$ from a menu $M \in \mathcal{M}$.

Definition 3. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is single-dipped w.r.t. the linear order L if for every $P \in \mathcal{Q}$ and every triple $\{x, y, z\} \subseteq X$ with $z \in b(\{x, y, z\}, P)$:

$$[xLyLz \text{ or } zLyLx] \implies xPy.$$

In other words, among the alternatives positioned on either side of the bottom-ranked alternative, those farther away from it are more preferred. The following example contains two instances of RUMs which possess the single-peaked and single-dipped property depending on the linear order.

Example 1 *(continued).* Consider the supports $\mathcal{P}_\mu$ and $\mathcal{P}_{\mu'}$. Let the linear order be $wLxLyLz$. As illustrated in Fig. 1, the two supports are single-peaked with respect to the linear order L. Observe that for the linear order $xLwLzLy$, the middle alternative in every triple is never most preferred, and hence the supports of the RUMs are single-dipped on a line as illustrated in Fig. 2.

The above example contains two instances of single-peaked and single-dipped RUMs that are observationally equivalent. This is no coincidence as the two domains are RUM-exclusive as we show in our first main result.

Theorem 1. *The single-peaked and single-dipped on a line preference domains are RUM-exclusive.*

Proof. See Appendix A.1. □

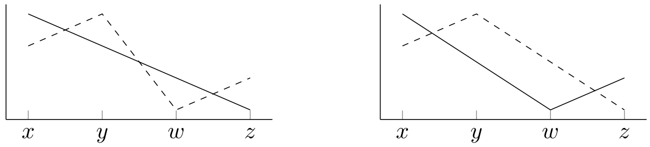

Fig. 3. Peak-pit on a line RUMs with supports $\mathcal{P}_\mu$ (left) and $\mathcal{P}_{\mu'}$ (right).

Intuitively, in single-peaked on a line RUMs, the middle alternative in any set of three alternatives, as per the linear order, is never the least preferred option within that set for all preferences in the support. If we suppose that there is another RUM that rationalizes the same stochastic choice function, but violates single-peakedness on a line, then for some triple the middle alternative is ranked last by some preference in the support. Therefore, the stochastic choice function generated by the non-single-peaked model has to assign a lower choice probability of the middle alternative from at least one binary menu. This means that the two RUMs do not generate the same choice function, which is a contradiction. The analogous argument holds for single-dipped on a line RUMs.

3.2. Single-crossing and peak-pit on a line domains

We now consider the single-crossing domain (Mirrlees, 1971; Spence, 1974). Let us first recall its definition for a given linear order of the alternatives.

Definition 4. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is single-crossing w.r.t. the linear order L if there exists a linear order L^* over $\mathcal{Q}$ such that for all pairs $\{x, y\} \in X$ with xLy and all $P, P' \in \mathcal{Q}$ with PL^*P':

$$xP'y \implies xPy.$$

The single-crossing condition implies that the preferences in the profile can be ordered unambiguously according to their compliance with the linear order L over the set of alternatives. For example, the preference profiles depicted in the left panels of Figs. 1 and 2 are single-crossing, whereas the profiles on the right are not.

Interestingly, Apesteguia et al. (2017) show that among the many rationalizing models, the single-crossing model is always unique, which directly leads to the following result.

Corollary 1. *The single-crossing preference domain is not RUM-exclusive.*

Thus, the preferences of the agent cannot be unambiguously revealed to be single-crossing, because of the multiplicity of representations. Given this negative result, we aim to find a superset of the single-crossing domain that has the RUM-exclusivity property and rationalizes the same choice datasets (see Section 4.2). We call this preference domain peak-pit on a line and define it formally below.

Definition 5. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is peak-pit w.r.t. the linear order L if for every $P \in \mathcal{Q}$ and every triple $\{x, y, z\} \subseteq X$ with $z \in t(\{x, y, z\}, P)$:

$$[xLyLz \text{ or } zLyLx] \implies yPx,$$

provided that $\exists P' \in \mathcal{Q}$ such that $y \in t(\{x, y, z\}, P')$.

Evidently, a preference profile is peak-pit on a line if the preference profile is either single-peaked or single-dipped with respect to L *on each triple* separately. An alternative way of defining this property is to say that a preference profile is peak-pit on a line if for each triple $xLyLz$ the middle alternative is never most and least preferred for some preferences over the triple, formally, $\nexists P, P' \in \mathcal{Q}$ such that $y \in t(\{x, y, z\}, P)$ and $y \in b(\{x, y, z\}, P')$. We illustrate the peak-pit on a line property with the RUMs given in Example 1.

Example 1 *(continued).* Consider the supports $\mathcal{P}_\mu$ and $\mathcal{P}_{\mu'}$. Let the linear order be $xLyLwLz$ as shown in Fig. 3. It can be easily seen that both preference profiles violate single-peakedness and single-dippedness for the given linear order, and preference P_2 is even neither single-peaked nor single-dipped. Both preference profiles satisfy the peak-pit on a line condition because for each triple the middle alternative is never ranked top *and* bottom by the preferences in the profile.

The fact that the peak-pit on a line domain combines characteristics of the single-peaked and single-dipped domains implies that our previous exclusiveness result extends naturally to peak-pit on a line preferences. Specifically, if a choice function is consistent with a model such that the preferences are single-peaked (single-dipped) on an arbitrary triple, then the supports of all rationalizing models are single-peaked (single-dipped) on that triple, hence peak-pit on a line, which leads to our next main result.

Theorem 2. *The peak-pit on a line domain is RUM-exclusive.*

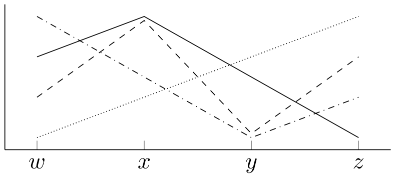

Fig. 4. A peak-pit on a line preference profile that is neither single-peaked nor single-dipped nor single-crossing for any linear order.

Proof. See Appendix A.2. □

To the best of our knowledge, peak-pit on a line is a novel preference property. Its applicability lies in its ability to accommodate double-(or even multiple) peaked preferences and simultaneously to have a unique majority voting equilibrium (see Section 3.3). Double-peaked preferences are potentially relevant in political contexts as a result of voter uncertainly about the positioning of the candidates on a line (Potthoff and Munger, 2005) and as a preference for "do something" in politics (Davis et al., 1970; Egan, 2014), which captures the consensus in a society that the current status quo policy is ineffective and some action needs to be taken.[7] They also arise naturally when alternatives are multidimensional (Cooter, 2000) and in facility location problems (Filos-Ratsikas et al., 2017).[8]

Unlike single-peakedness and single-dippedness, the peak-pit property relies on the interdependencies among the preferences in the profile. It allows for more flexibility of the preferences than single-peakedness if there is more consensus in the society about the best or worst alternatives available. In the special case in which each alternative in X is a greatest element for at least one preference in the profile, peak-pit preferences satisfy single-peakedness. On the contrary, if all alternatives in X are bottom ranked by the preferences in the profile, the peak-pit domain coincides with the single-dipped one. Thus, the peak-pit on a line condition is a generalization of single-peakedness and single-dippedness on a line. As stated in the result below, it encompasses single-crossing as well.

Corollary 2. *If a preference profile is single-peaked on a line, single-dipped on a line, or single-crossing, then it is also peak-pit on a line.*

Proof. See Appendix A.3. □

The fact that the peak-pit on a line condition is implied by both single-peakedness on a line and single-crossing is an interesting feature of this domain, because single-peakedness on a line and single-crossing are otherwise independent from each other, meaning that there exist preference profiles which are single-peaked on a line but not single-crossing and vice versa. We provide an example of a preference profile below which is peak-pit on a line, but neither single-peaked nor single-dipped nor single-crossing for any linear order, thus showing that the peak-pit domain is more general than the union of the three smaller domains.

Example 2. Consider a set of alternatives $X = \{w, x, y, z\}$ and a preference profile $Q = \{P_1, P_2, P_3, P_4\}$ with $wP_1xP_1zP_1y$, $xP_2wP_2yP_2z$, $xP_3zP_3wP_3y$, and $zP_4yP_4xP_4w$. The profile is not single-peaked for any linear order. This is the case because each of the alternatives $\{x, y, z\}$ is bottom ranked from the triple for at least one preference in Q, hence there is no L over which the preferences are single-peaked over the triple. Similarly, the profile is not single-dipped on a line as well since each of the alternatives in $\{w, x, z\}$ is top ranked for at least one preference in Q.

Moreover, the profile is also not single-crossing for any linear order. Assume by contradiction that it is single-crossing and $P_1L^*P_2$. Single-crossing implies that the linear order over X is such that wLx and zLy, which in turn implies that $P_1L^*P_4L^*P_2$. It follows again by single-crossing that if wLy then $P_2L^*P_4$, and if yLw then $P_4L^*P_1$, both leading to contradiction. An analogous contradiction can be found if $P_2L^*P_1$.

For the linear order $wLxLyLz$ depicted in Fig. 4, the profile is peak-pit, because the preferences are single-dipped only over the triples, for which the middle alternative is not the greatest element from the respective triple for any of the preferences in Q. Preferences P_1 and P_3 can be interpreted as an instance of the so called "do something" preferences (Egan, 2014) for the given linear order and status quo y, while P_2 and P_4 represent the classical single-peaked preferences.

[7] See Example 2 for an abstract illustration.

[8] Peak-pit preferences accommodate not only double-peaked, but also double-crossing preferences. In order to observe that in the facility location setting, suppose that a shop is located on a street and although the residents prefer to live in close proximity to it along the street, they would also dislike to live exactly where it is positioned, because of the increased noise and traffic. This leads to the emergence of double-peaked preferences. If this trade-off impacts agents' preferences differently, so that some agents' local peaks are closer, others are farther from the shop's location, this would create an instance of multiple-crossing preferences. Such preference profiles are also peak-pit on a line.

3.3. Locally peak-pit, triple-wise value-restricted and peak-monotone domains

Next, we study whether the RUM-exclusiveness property can be extended to larger existing preference domains that nest peak-pit on a line preferences. These domains include Danilov et al. (2012)'s (local) peak-pit or tiling domain, Sen (1966)'s triple-wise value-restriction, and Barberà and Moreno (2011)'s peak-monotonicity. We first provide the formal definitions of the properties.

Recall that the top- and bottom-ranked alternative for a given triple $\{x, y, z\}$ and preference P are denoted by $t(\{x, y, z\}, P)$ and $b(\{x, y, z\}, P)$.

Definition 6. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is locally peak-pit if for every triple $\{x, y, z\} \subseteq X$ there is no $P \in \mathcal{Q}$ such that $x \in t(\{x, y, z\}, P)$ or there is no $P \in \mathcal{Q}$ such that $x \in b(\{x, y, z\}, P)$.

Contrary to all domain restrictions previously analyzed in this paper, the local peak-pit domain is not defined for a particular linear order. In fact, local peak-pit preferences generalize local single-peaked preferences, where for every triple $\{x, y, z\} \subseteq X$, there is no $P \in \mathcal{Q}$ such that $x \in b(\{x, y, z\}, P)$ and local single-dipped preferences, where there is no $P \in \mathcal{Q}$ such that $x \in t(\{x, y, z\}, P)$.[9]

Let the middle-ranked alternative be denoted by $m(\{x, y, z\}, P)$. Triple-wise value-restriction is defined formally below.

Definition 7. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is triple-wise value-restricted if for every triple $\{x, y, z\} \subseteq X$ there is no $P \in \mathcal{Q}$ such that $x \in t(\{x, y, z\}, P)$ or there is no $P \in \mathcal{Q}$ such that $x \in m(\{x, y, z\}, P)$ or there is no $P \in \mathcal{Q}$ such that $x \in b(\{x, y, z\}, P)$.

Similarly to the local peak-pit domain, triple-wise value-restriction generalizes local single-peakedness and single-dippedness, but also group-separability (Inada, 1964), where there is no $P \in \mathcal{Q}$ such that $x \in m(\{x, y, z\}, P)$.

Barberà and Moreno (2011) define the peak-monotone domain and show that it is a common root for single-peakedness on a line and single-crossing, similarly to the peak-pit on a line domain.[10]

Definition 8. A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is peak-monotone w.r.t. the linear order L if for every $P, P' \in \mathcal{Q}$ and every triple $\{x, y, z\} \subseteq X$ with $z \in t(\{x, y, z\}, P)$ and $z \in t(X, P')$:

$$[xLyLz \text{ or } zLyLx] \implies yPx,$$

provided that $\exists P'', P''' \in \mathcal{Q}$ such that $y \in t(X, P'')$ and $x \in t(X, P''')$, and

$$[xLyLz \text{ or } zLyLx] \implies yP'x,$$

provided that $\exists P'' \in \mathcal{Q}$ such that $y \in t(X, P'')$.

Loosely speaking, peak-monotonicity requires that the preferences are single-peaked on a line only over the most preferred alternatives in the profile. Barberà and Moreno (2011) show that such preferences arise naturally in models involving choices between public and private provision of a public good (Epple and Romano, 1996; Stiglitz, 1974). Note that peak-monotonicity and local peak-pit (and thus triple-wise value-restriction) are independent properties as illustrated in the example below.

Example 3. Consider a set $X = \{w, x, y, z\}$ and a preference profile $\mathcal{Q} = \{P_1, P_2, P_3\}$ with $wP_1xP_1yP_1z$, $wP_2yP_2zP_2x$, and $wP_3zP_3xP_3y$. Since all preferences $P \in \mathcal{Q}$ have the same greatest element $w \in t(X, P)$ the profile trivially satisfies peak-monotonicity for any linear order. However, it violates triple-wise value-restriction, and thus the peak-pit condition, as the preferences over the triple $\{x, y, z\}$ rank each of the alternatives as top, middle and bottom for some preference in the profile.

To show that the converse can also hold, consider a profile $\mathcal{Q} = \{P_1, P_2, P_3, P_4\}$ with $wP_1xP_1yP_1z$, $xP_2wP_2yP_2z$, $yP_3wP_3xP_3z$, and $zP_4wP_4xP_4y$. This profile is locally single-peaked and thus locally peak-pit and triple-wise value-restricted since w is never bottom-ranked from any triple containing it and x is never bottom-ranked from $\{x, y, z\}$. However, it is not peak-monotone for any linear order. Note that all alternatives in X are ranked top by at least one preference in the profile, hence peak-monotonicity coincides with single-peakedness on a line. Since w is never bottom-ranked from any triple, the linear order should be such that w is simultaneously positioned between x and y, y and z, and x and z, which is impossible. Thus, the preference profile does not satisfy single-peakedness on a line and peak-monotonicity for any linear order.

Although the local peak-pit and triple-wise value-restricted domains are independent from peak-monotonicity, all three domains are RUM-exclusive.

Theorem 3. *The local peak-pit, the triple-wise value-restricted, and the peak-monotone domains are RUM-exclusive.*

[9] Local single-peakedness is also known as single-peakedness over triples or Arrowian single-peakedness.

[10] Barberà and Moreno (2011, Def. 12)'s definition of top-monotonicity is adapted here for strict preferences.

Proof. See Appendix A.4. □

In the following corollary, we show that local peak-pit, peak-monotonicity and triple-wise value-restriction are weaker properties than peak-pit on a line.

Corollary 3. *If a preference profile is peak-pit on a line, then it is also locally peak-pit, triple-wise value-restricted, and peak-monotone.*

The proof of the statement follows directly from the definitions. The definition of peak-pit on a line preferences states that the *middle* alternative from any triple according to a global linear order is either not top or not bottom ranked by the preferences in the profile, whereas triple-wise value-restriction holds if some alternative from each triple is not top, not middle or not bottom ranked by all preferences in the profile.

Further, the peak-pit on a line property requires that single-peakedness has to hold whenever there exists $P \in Q$ such that $y \in t(\{x, y, z\}, P)$ instead of peak-monotonicity's more restrictive $y \in t(X, P)$ and its additional restriction on the extreme alternatives according to L in the triple.

Note that the relationship between peak-pit on a line and local peak-pit domains is analogous to the one between single-peaked on a line and local single-peaked domains. Specifically, when the local peak-pit domain is of maximal width, i.e. it contains two completely reversed preferences, it belongs to the peak-pit on a line domain restriction.[11]

It follows immediately from Proposition 3 that the peak-pit on a line domain inherits several desirable properties known to hold for triple-wise value-restricted and peak-monotone preference domains from the social choice literature. Notably, Sen (1966) shows that the social welfare function constructed from pairwise majority voting over triple-wise value-restricted domains satisfies all of Arrow (1951)'s conditions, in particular that it is transitive. Furthermore, Barberà and Moreno (2011) prove that peak-monotonicity guarantees the existence of Condorcet winners for a wide class of aggregation rules, including the majority rule. Peak-monotonicity also preserves a version of the median voter result. Finally, peak-pit domains of maximal width have convenient geometric representations as rhombus tilings (Danilov et al., 2012), distributive lattices (Puppe and Slinko, 2019), and arrangements of pseudolines (Galambos and Reiner, 2008).

4. Revealing preference domains

In terms of revealing preference domains from random choice, the RUM-exclusiveness property means that if we can find a random utility model (RUM) rationalizing the choice data and belonging to a specific exclusive domain, the preferences are revealed to belong to that domain. This section examines the properties of the stochastic choice function corresponding to the studied model subclasses. Consequently, we can directly reveal the preference domain from the choice function without the need to find a rationalizing RUM.

A well-known necessary but not sufficient condition for rationalizability when the set of alternatives consists of four or more elements is regularity (Block and Marschak, 1960). It asserts that the probability of choosing an alternative does not increase when the menu is expanded.

Definition 9. A stochastic choice function ρ satisfies regularity (REG) if for all $M, M' \in \mathcal{M}$ with $M' \subseteq M$ it holds that $\rho(x, M) \leqslant \rho(x, M')$.

Although this choice axiom is not specific to any single model subclass that we studied so far, it is crucial for ensuring rationalizability in several forthcoming results.

4.1. Single-peaked and single-dipped RUMs

Single-peaked and single-dipped on a line RUMs have been characterized by Apesteguia et al. (2017) in terms of the above defined regularity, and their strong centrality (SCEN) and extremality (EXT) preference properties, which we state formally below.

Definition 10. A stochastic choice function ρ satisfies strong centrality (SCEN) if for every triple $xLyLz$ it holds that $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$.

SCEN requires that in every triple of alternatives the probability of choosing one extreme alternative is not affected by the presence of the other extreme alternative. The intuition for such choice behavior is that all positive features of the other extreme alternative are incorporated in the middle alternative, thus making the choice of the extreme alternative unattractive.

Extremality requires that for each triple, the middle alternative is never chosen from the triple and is reminiscent of the extremeness axiom by Gul and Pesendorfer (2006).

Definition 11. A stochastic choice function ρ satisfies extremality (EXT) if for every triple $xLyLz$ it holds that $\rho(y, \{x, y, z\}) = 0$.

[11] I thank Clemens Puppe for this valuable insight.

Apesteguia et al. (2017)'s characterization result reads that single-peaked RUMs generate stochastic choice functions that are REG and SCEN, and for each stochastic choice function satisfying REG and SCEN there exists a single-peaked RUM that rationalizes it. The analogous result holds for single-dipped RUMs and choice functions satisfying EXT. The RUM-exclusiveness property of the single-peaked and single-dipped domains given in Theorem 1 can be used to extend Apesteguia et al. (2017)'s results to all multiple RUM representations.

Corollary 4. *A stochastic choice function ρ satisfies REG and SCEN (EXT) if and only if it is rationalizable only by single-peaked on a line (single-dipped on a line) RUMs. Moreover, when the stochastic choice function is generated by a single-peaked RUM, it holds for all subsets $M = \{x_1, \dots, x_{|M|}\} \subseteq X$ with $|M| \geqslant 3$ and $x_1 L x_2 L \dots L x_{|M|}$ that*

$$\rho(x_1, M) = \rho(x_1, \{x_1, x_2\}), \rho(x_{|M|}, M) = \rho(x_{|M|}, \{x_{|M|-1}, x_{|M|}\}), \text{ and}$$
$$\rho(x_i, M) = \rho(x_i, \{x_{i-1}, x_i\}) - \rho(x_{i+1}, \{x_i, x_{i+1}\}), \forall i \in [2, |M| - 1].$$

When the stochastic choice function is generated by a single-dipped RUM it holds for all such subsets $M \subseteq X$ that

$$\rho(x_1, M) = \rho(x_1, \{x_1, x_{|M|}\}), \rho(x_{|M|}, M) = \rho(x_{|M|}, \{x_1, x_{|M|}\}), \text{ and}$$
$$\rho(x_i, M) = 0, \forall i \in [2, |M| - 1].$$

Proof. See Appendix A.5. □

Therefore, choice frequencies reveal precisely whether the underlying preferences are single-peaked (single-dipped) or not. This is because by showing that all rationalizing models belong to the same class of single-peaked RUMs, we know that the true generating model has to be single-peaked as well.[12]

Combining domain exclusiveness with characterization results of the respective RUM subclasses can be used for choice prediction after a perturbation of the state probabilities. More specifically, suppose that we observe a decision maker whose choices satisfy, say REG and SCEN. If the states are fixed and there is a change in their probability of occurrence, the choice frequencies of the agent would still satisfy the same choice properties because the preferences of the agent are revealed to be single-peaked. Furthermore, as shown in Corollary 4, we only need to observe the choice frequencies from binary sets in order to recover the whole new stochastic choice function. We illustrate this point with the following example.

Example 1 *(continued)*. Consider again the stochastic choice functions $\rho(x, M) = \rho'(x, M)$ for all $x \in M$ and $M \in \mathcal{M}$. It can be easily verified that ρ satisfies REG and SCEN for the linear order $wLxLyLz$, for example $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\}) = 0.5$. Thanks to Apesteguia et al. (2017, Corollary 1), we can conclude that ρ can be rationalized by a single-peaked RUM, say μ. Corollary 4 implies that all other RUMs rationalizing ρ, such as μ', are also single-peaked with respect to L, thus the preferences of the agent are revealed to be single-peaked.

Suppose now that there is a change in the state probabilities and we observe that the new choice frequency is $\rho^*(x, \{w, x\}) = 1$, $\rho^*(y, \{x, y\}) = 0.2$ and $\rho^*(w, \{w, z\}) = 0.2$. Although the new stochastic choice function is not consistent with a RUM with support $\mathcal{P}_\mu$, we can use Corollary 4 to recover the whole new stochastic choice function, for example $\rho^*(x, \{w, x, y, z\}) = 0.8$.

4.2. Peak-pit RUMs

In their main theorem, Apesteguia et al. (2017) characterize single-crossing RUMs in terms of REG and a property called centrality, which encompasses SCEN and EXT.

Definition 12. A stochastic choice function ρ satisfies centrality (CEN) if for every triple $xLyLz$ with $\rho(y, \{x, y, z\}) > 0$ it holds that $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$.

Our next result establishes that the preferences of an agent whose choices satisfy REG and CEN are revealed to be peak-pit on a line.

Proposition 1. *A stochastic choice function ρ satisfies REG and CEN if and only if it is rationalizable only by peak-pit on a line RUMs.*

Proof. See Appendix A.6. □

Apesteguia et al. (2017) discover an interesting relationship between stochastic choice functions generated by single-peaked on a line and single-crossing RUMs, namely that the former are a subclass of the latter, although the properties of the supports are not

[12] Note that this result holds only for RUMs over strict preferences and that single-peakedness over weak preferences cannot be inferred from choice data. However, single-peaked RUMs over weak preferences generate stochastic choices that satisfy regularity and strong centrality. More details are available upon request.

otherwise related. Corollary 4 and Proposition 1 contribute to the better understanding of this seeming inconsistency. Since single-peaked on a line RUMs are a subclass of peak-pit on a line RUMs, the induced stochastic choice functions preserve this relationship.

Proposition 1 can also be used for choice prediction as a result of a shift in the distribution over the states. If a researcher observes a REG and CEN stochastic choice function, the preferences of the agent are revealed to be peak-pit on a line. Under a different state distribution, the choice function still has to satisfy the same properties, thus making some choice experiments unnecessary. Moreover, similarly to the case with single-peaked and single-dipped RUMs, a researcher can recover the whole new stochastic choice function corresponding to a peak-pit model from binary choices only. More details are provided in Appendix A.7.

Recall that the local peak-pit condition encompasses local single-peakedness and local single-dippedness. In Section 3.1 we showed that single-peaked on a line and single-dipped on a line RUMs can be characterized using SCEN and EXT. The choice properties corresponding to the local versions of single-peakedness and single-dippedness can be defined analogously over triples without assuming a global linear order i.e. a stochastic choice function ρ satisfies local centrality (L-CEN) if for every triple $\{x, y, z\} \subseteq X$ with $\rho(y, \{x, y, z\}) > 0$, it holds that $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$. As we show below, the local peak-pit condition corresponds to L-CEN stochastic choice functions.

Proposition 2. *A rationalizable stochastic choice function ρ satisfies L-CEN if and only if it is rationalizable only by locally peak-pit RUMs.*

Proof. See Appendix A.8. □

The L-CEN property is reminiscent of the Independence axiom by Masatlioglu and Vu (2023), in a sense that both conditions require that choice probabilities remain unchanged when a certain alternative is removed from the choice set. There are important differences, however, such as L-CEN requiring that choice probabilities remain constant for at least two alternatives in the profile, while Independence requiring only one. On the other hand, Independence specifies for which alternatives in the triple the choice probabilities are robust to specific changes in the choice set, while L-CEN does not.

4.3. Triple-wise value-restricted random utility models

In order to characterize triple-wise value-restriction, we need to define additionally a choice property that we call local independence of similar alternatives, corresponding to the locally group-separable preferences already mentioned in Section 3.3.

Definition 13. A stochastic choice function ρ satisfies local independence of similar alternatives (L-ISA) if for every triple $\{x, y, z\} \subseteq X$, it holds that $\rho(x, \{x, y\}) = \rho(x, \{x, z\}) = \rho(x, \{x, y, z\})$.

Intuitively, such stochastic choice functions reveal that alternatives y and z act as duplicates or substitutes compared to x, because the presence of one or the other in the menu does not affect the choice probability of x. Such choice behavior is to be expected in analogous situations to Debreu (1960)'s famous red bus/blue bus example.

As we show below, stochastic choices that satisfy L-CEN or L-ISA reveal that the underlying preferences are triple-wise value-restricted.

Proposition 3. *A rationalizable stochastic choice function ρ satisfies L-CEN or L-ISA if and only if it is rationalizable only by triple-wise value-restricted RUMs.*

Proof. See Appendix A.9. □

Although the class of triple-wise value-restricted RUMs is the largest that we analyze, the restrictive properties that the choice function needs to satisfy might raise concerns about its practical relevance. However, the following observation offers a reason for optimism: as we will show, these models satisfy a form of stochastic transitivity, which has been observed in laboratory experiments. In the stochastic choice literature, there are different degrees of transitivity of choices (Luce and Suppes, 1965). Whenever $\min\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\} \geqslant \frac{1}{2}$ and $\max\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\} > \frac{1}{2}$, the choice function satisfies weak stochastic transitivity if $\rho(x, \{x, z\}) \geqslant \frac{1}{2}$, moderate stochastic transitivity if $\rho(x, \{x, z\}) \geqslant \min\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\}$, and strong stochastic transitivity if $\rho(x, \{x, z\}) \geqslant \max\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\}$ for all such triples.[13] The fact that no Condorcet cycles can arise on triple-wise value-restricted preference profiles translates in the random choice framework as weak stochastic transitivity of the generated choice function. Below we prove that L-CEN and L-ISA choice functions satisfy an even stronger notion of transitivity.

Proposition 4. *Triple-wise value-restricted RUMs generate stochastic choice functions that satisfy moderate, but not strong stochastic transitivity.*

[13] The classical definitions of stochastic transitivity do not assume our additional premise $\max\{\rho(x, \{x, y\}), \rho(y, \{y, z\})\} > \frac{1}{2}$.

Proof. See Appendix A.10. □

Stochastic choice functions that satisfy moderate, but not strong stochastic transitivity are well-documented in the experimental literature (Luce, 1977; Luce and Suppes, 1965; Switalski, 2001; Tversky, 1972; Tversky and Russo, 1969). Thus, the experimental evidence does not contradict stochastic choices resulting from triple-wise value-restricted RUMs.

4.4. Peak-monotone random utility models

In order to find the property of the stochastic choice function corresponding to peak-monotone RUMs, we need to relax the definition of CEN appropriately. We define weak centrality as shown below.

Definition 14. A stochastic choice function ρ satisfies weak centrality (WCEN) if for every triple $xLyLz$ or $zLyLx$ with $\rho(y,X)>0$ and $\rho(x,X)>0$ it holds that $\rho(x,\{x,y\})=\rho(x,\{x,y,z\})$ and $\rho(z,\{y,z\})=\rho(z,\{x,y,z\})$.

Note that WCEN is indeed a weaker version of CEN in the presence of REG, because $\rho(y,X)>0$ implies $\rho(y,\{x,y,z\})>0$ for any triple $\{x,y,z\}\subseteq X$ and it also requires at least one of the extreme alternatives to be minimally attractive. As we show below, WCEN reveals that the generating model is peak-monotone.

Proposition 5. *A rationalizable stochastic choice function ρ satisfies WCEN if and only if it is rationalizable only by peak-monotone RUMs.*

Proof. See Appendix A.11. □

Note that in Propositions 2, 3 and 5 we assume rationalizability of the stochastic choice function. The reason is that even if we assume REG in addition, rationalizability is still not guaranteed. Thus, the stronger restrictions on the stochastic choice function needed for the main characterization results of single-peaked, single-dipped, and peak-pit on a line RUMs, and in particular the requirement that the properties hold with respect to a given linear order, cannot be easily relaxed without giving up rationalizability. The following example illustrates this point.

Example 4. Consider a set $X=\{w,x,y,z\}$ and $M\in\mathcal{M}$. The stochastic choice function is defined such that $\rho(i,M)=0.5$ when $|M|=2$ and $i\in M$. When $|M|=3$, we have $\rho(w,M)=0$ and $\rho(y,M)=\rho(z,M)=0.5$ whenever the corresponding alternative is in the menu. We assume further $\rho(w,X)=\rho(x,X)=0$ and $\rho(z,X)=\rho(y,X)=0.5$.
REG is satisfied because the probability of choosing an alternative from any triple lies between the choice probability from sets with two or four alternatives. Observe that L-CEN is satisfied, because $\rho(w,\{w,x,y\})=\rho(w,\{w,y,z\})=0$ and $\rho(w,\{w,x,z\})=\rho(x,\{x,y,z\})=0$. Furthermore, the stochastic choice function fulfills WCEN as well. Since only $\rho(y,X)>0$ and $\rho(z,X)>0$, we can let $yLwLxLz$ so that the middle alternative in any triple is never chosen from X. WCEN is trivially satisfied for L. This stochastic choice function is, however, not rationalizable.[14]

Nevertheless, one could check that a stochastic choice function is rationalizable by a general RUM using the characterization result of Falmagne (1978) and Barberà and Pattanaik (1986) and check for L-CEN, L-ISA, or WCEN to reveal the corresponding property of the underlying RUM.

5. Discussion

In this paper we study the RUM-exclusiveness of various well-known preference domains and how to infer them from stochastic choice data. We establish that single-peaked, single-dipped, and a novel more general preference domain called peak-pit on a line are RUM-exclusive, as opposed to the domain of single-crossing preferences. We also show that the peak-pit on a line domain is nested in the domains of triple-wise value-restricted and peak-monotone preferences and demonstrate that these properties can be revealed from choice frequencies as well.

For most of our results we assume an underlying linear order over the alternatives. Although there are many contexts in which a linear order is suggested by the nature of the alternatives in the set (as for example in the cases when the alternatives are numbers arranged in an order, political parties ordered on the political spectrum, etc.), there might be others in which deducing the linear order is less intuitive. Ballester and Haeringer (2011), Bredereck et al. (2013) and Puppe (2018) show that single-peakedness on a line and single-crossing can be defined without the use of linear orders by identifying specific forbidden substructures. An analogous approach could be used to redefine the properties of stochastic choice functions used in this paper to avoid the assumption of a linear order. This will potentially simplify the computational problem of verifying whether a stochastic choice function possesses a particular property.

[14] In order to verify that the stochastic choice function is not rationalizable, we need to find a Block-Marschak polynomial (Block and Marschak, 1960) which is negative, for example: $BM(x,\{w,x\},\{w,x,y,z\})=\rho(x,\{w,x\})-\rho(x,\{w,x,y\})-\rho(x,\{w,x,z\})+\rho(x,\{w,x,y,z\})<0$. This implies that the stochastic choice function is not rationalizable (Barberà and Pattanaik, 1986; Falmagne, 1978).

Furthermore, this paper focuses on preference domains that satisfy a property defined on triples and with respect to a linear order. Those features seem to be crucial for RUM-exclusiveness as long as we do not assume an additional structure over the preferences in a profile, like in the case with single-crossing preferences. Future research could investigate the robustness of the RUM-exclusiveness property under alternative relationships between alternatives, such as an order on a circle (Peters and Lackner, 2020) or a tree (Demange, 1982).

The new peak-pit on a line preference domain sets a direction for future research in social choice. Related to the above point, it will be useful to define the property without relying on a linear order. Further, since the peak-pit on a line domain is more general than the union of single-peaked, single-dipped and single-crossing preference profiles, the results on strategy-proofness known to hold on these domains (Alcalde–Unzu et al., 2023; Moulin, 1980; Saporiti and Tohmé, 2006) could potentially be extended to peak-pit on a line preference profiles. Finally, finding a utility representation of peak-pit preferences and identifying parametric families of utilities that are peak-pit, but neither single-peaked nor single-dipped, would ease the applicability of the preference domain in other economic models.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Proof of Theorem 1

We first prove the RUM-exclusiveness of single-peaked preferences. Consider a stochastic choice function ρ which is rationalizable with a single-peaked RUM μ_1. Let μ_2 be another RUM that rationalizes ρ but violates single-peakedness. Therefore, there exists a preference $P^* \in \mathcal{P}_{\mu_2}$ over a triple $\{x, y, z\} \subseteq X$ that ranks zP^*xP^*y and either $xLyLz$ or $zLyLx$. Since $\mathcal{P}_{\mu_1}$ is single-peaked, it holds for all $P \in \mathcal{P}_{\mu_1}$ with $z \in t(\{x, y, z\}, P)$ that yPx whenever $xLyLz$ or $zLyLx$. This means that $\sum_{P\in\mathcal{P}:zPxPy} \mu_1(P) = 0$ and $\sum_{P\in\mathcal{P}:zPxPy} \mu_2(P) > 0$.

The following lemma shows the relationship between a rationalizable stochastic choice function over a triple and its RUM representation. It follows directly from the definition of a rationalizable stochastic choice function.

Lemma 1. *Consider a triple $\{x, y, z\} \subseteq X$. If ρ is rationalizable with some RUM μ, then $\rho(y, \{y, z\}) - \rho(y, \{x, y, z\}) = \sum_{P\in\mathcal{P}:xPyPz} \mu(P)$.*

It follows from Lemma 1 that $\sum_{P\in\mathcal{P}:zPxPy} \mu_1(P) = \rho(x, \{x, y\}) - \rho(x, \{x, y, z\}) = 0$. Since the two RUMs satisfy the same stochastic choice function, we must have $\sum_{P\in\mathcal{P}:zPxPy} \mu_2(P) = 0$ as well, which is a contradiction. Therefore, $\mathcal{P}_{\mu_2}$ is single-peaked too.

Next, we show the RUM-exclusiveness property of the single-dipped domain. Consider a stochastic choice function ρ which is rationalizable with a single-dipped RUM μ_1. We denote with μ_2 another RUM that rationalizes ρ but violates single-dippedness. Thus, there is a preference in $P^* \in \mathcal{P}_{\mu_2}$ and a triple $\{x, y, z\} \subseteq X$ with either $xLyLz$ or $zLyLx$ and yP^*xP^*z. For the single-dipped RUM μ_1 it holds for all $P \in \mathcal{P}_{\mu_1}$ with $z \in b(\{x, y, z\}, P)$ that xPy whenever $xLyLz$ or $zLyLx$. This implies that $\sum_{P\in\mathcal{P}:yPxPz} \mu_1(P) = 0$ and $\sum_{P\in\mathcal{P}:yPxPz} \mu_2(P) > 0$.

Applying Lemma 1, we have that $\sum_{P\in\mathcal{P}:yPxPz} \mu_1(P) = \rho(x, \{x, z\}) - \rho(x, \{x, y, z\}) = 0$. Therefore, $\sum_{P\in\mathcal{P}:yPxPz} \mu_2(P) > 0$ is a contradiction. Consequently, $\mathcal{P}_{\mu_2}$ has to be single-dipped on a line as well.

A.2. Proof of Theorem 2

Consider a stochastic choice function ρ which is rationalizable with a peak-pit on a line RUM μ_1, i.e. the support $\mathcal{P}_{\mu_1}$ is such that for each triple the profile is either single-peaked or single-dipped w.r.t. the same linear order. Let μ_2 be another RUM that rationalizes ρ. We need to show that μ_2 is peak-pit on a line as well. Consider an arbitrary triple $xLyLz$. Suppose that $\mathcal{P}_{\mu_1}$ is single-peaked over that triple. Using our proof of Theorem 1, it follows that $\mathcal{P}_{\mu_2}$ is single-peaked over the triple with respect the same linear order as well. The analogous argument holds whenever $\mathcal{P}_{\mu_1}$ is single-dipped over the triple. Therefore, $\mathcal{P}_{\mu_2}$ can either be single-peaked or single-dipped over the triple, and therefore, peak-pit on a line.

A.3. Proof of Corollary 2

The statement for single-peaked on a line and single-dipped on a line preference profiles follows directly from the definitions. Let $\mathcal{Q} \subseteq \mathcal{P}$ be a single-crossing preference profile with a linear order over the preferences in $\mathcal{Q}$ given by L^*. Take a triple $xLyLz$ and assume by contradiction that $\mathcal{Q}$ violates peak-pit on a line, i.e. $\exists P, P' \in \mathcal{Q}$ such that $t(\{x, y, z\}, P) = z$ and $t(\{x, y, z\}, P') = y$, but xPy. Hence, $b(\{x, y, z\}, P) = y$. We can either have that (a) PL^*P' or (b) $P'L^*P$. Consider case (a) first. From zPy and the single-crossing property follows that $zP'y$. Then, however, $t(\{x,y,z\},P') \neq y$, which is a contradiction. Consider case (b). Since $yP'x$, then yPx again by single-crossing, violating $b(\{x,y,z\},P) = y$. Therefore, $\mathcal{Q}$ has to be peak-pit on a line.

A.4. Proof of Theorem 3

Let ρ be a rationalizable stochastic choice function by two RUMs μ_1 and μ_2 and $\{x,y,z\} \subseteq X$ be an arbitrary triple.

We first prove RUM-exclusiveness for peak-pit preferences. Suppose that μ_1 is peak-pit, thus either there is no $P \in \mathcal{P}_{\mu_1}$ such that $x \in t(\{x,y,z\},P)$ or there is no $P \in \mathcal{P}_{\mu_1}$ such that $x \in b(\{x,y,z\},P)$. Suppose first that the former statement is true, i.e. $\nexists P \in \mathcal{P}_{\mu_1}$ such that $xPyPz$ or $xPzPy$. Therefore, the support is single-dipped on that triple w.r.t. the linear orders $yLxLz$ and $zLxLy$. Since single-dippedness is a RUM-exclusive domain as we showed in Theorem 1, $\mathcal{P}_{\mu_2}$ has to be single-dipped w.r.t. the same linear orders over the triple, hence peak-pit over the triple. Analogously, if there is no $P \in \mathcal{P}_{\mu_1}$ such that $x \in b(\{x,y,z\},P)$, $\mathcal{P}_{\mu_1}$ is single-peaked on that triple w.r.t. the linear orders $yLxLz$ and $zLxLy$. Again by Theorem 1, $\mathcal{P}_{\mu_2}$ has to be single-peaked w.r.t. the same linear orders over the triple, hence peak-pit over the triple. Since the linear orders over the triples do not need to form a global linear order, we have shown that $\mathcal{P}_{\mu_2}$ is locally peak-pit.

Next, we prove that the triple-wise value-restricted domain is RUM-exclusive. Assume that $\mathcal{P}_{\mu_1}$ is triple-wise value-restricted, thus the profile is either peak-pit or group-separable over $\{x,y,z\}$. We have already shown that if it is peak-pit, then $\mathcal{P}_{\mu_2}$ is also peak-pit on the triple. We now assume that $\mathcal{P}_{\mu_1}$ is group-separable, that is there is no $P \in \mathcal{P}_{\mu_1}$ such that $x \in m(\{x,y,z\},P)$. This means that there is no $\nexists P \in \mathcal{P}_{\mu_1}$ such that $yPxPz$ or $zPxPy$. We use Lemma 1 from Appendix A.1 to show that the stochastic choice function must be such that $\rho(x,\{x,y\}) - \rho(x,\{x,y,z\}) = \rho(x,\{x,z\}) - \rho(x,\{x,y,z\}) = 0$. This implies that $\mathcal{P}_{\mu_2}$ does not contain the preferences $yPxPz$ or $zPxPy$ and is thus group-separable on the triple. Therefore, $\mathcal{P}_{\mu_2}$ is also triple-wise value-restricted.

Before we prove the RUM-exclusiveness of the peak-monotone domain, we provide an alternative definition and show the equivalence in the lemma below. Let $T_{\mathcal{Q}} = \bigcup_{P\in\mathcal{Q}} t(X,P)$ be a set containing all peaks in a preference profile.

Lemma 2. *A preference profile $\mathcal{Q} \subseteq \mathcal{P}$ is peak-monotone if and only if it holds for all triples $xLyLz$ or $zLyLx$ with $\{x,y\} \subseteq T_{\mathcal{Q}}$ that there is no $P \in \mathcal{Q}$ such that $xPzPy$ or $zPxPy$.*

Proof. Consider a peak-monotone profile $\mathcal{Q}$ and assume w.l.o.g. that $xLyLz$ and $\{x,y\} \subseteq T_{\mathcal{Q}}$. Since $\{x,y\} \subseteq T_{\mathcal{Q}}$, it follows from Definition 8 that $\nexists P' \in \mathcal{Q}$ such that $zP'xP'y$. Assume by contradiction that $\exists P \in \mathcal{Q}$ such that $xPzPy$. Peak-monotonicity implies that $x \notin t(X,P)$. Denote $w \in t(X,P)$. We distinguish two cases according to the position of w in the linear order. Consider first the case wLy and the triple $\{w,y,z\}$. Peak-monotonicity implies that $wPyPz$ because $\{w,y\} \subseteq T_{\mathcal{Q}}$, which is a contradiction to $xPzPy$. Then, it must be that yLw. Again peak-monotonicity implies that $wPyPx$, which is a contradiction to $xPzPy$. Therefore, there is no $P \in \mathcal{Q}$ for which $xPzPy$.

We will now show that the converse holds as well. Consider a profile $\mathcal{Q}$ for which it holds for all triples $xLyLz$ or $zLyLx$ with $\{x,y\} \subseteq T_{\mathcal{Q}}$ there is no $P \in \mathcal{Q}$ for which $xPzPy$ or $zPxPy$. Take a triple $xLyLz$ for which it holds that $\exists P''$ such that $y \in t(X,P'')$. Consider an arbitrary $P' \in \mathcal{Q}$ with $z \in t(X,P')$. Since $\{z,y\} \subseteq T_{\mathcal{Q}}$, there is no preference $P \in \mathcal{Q}$ for which $zPxPy$, and hence we have $zP'yP'x$ and peak-monotonicity is satisfied. Next, suppose that $xLyLz$ is such that $x \in t(X,P)$ for some $P \in \mathcal{Q}$. Consider an arbitrary $P' \in \mathcal{Q}$ with $z \in t(\{x,y,z\},P')$. Since $\{x,y\} \subseteq T_{\mathcal{Q}}$ it holds that $zP'yP'x$ and peak-monotonicity is satisfied. □

In order to show the RUM-exclusiveness of peak-monotonicity, we assume that μ_1 is peak-monotone. We need to show that μ_2 is peak-monotone as well. Take a triple $xLyLz$ such that $\{x,y\} \subseteq T_{\mathcal{P}_1}$. Peak-monotonicity implies there is no $P \in \mathcal{P}_{\mu_1}$ such that $xPzPy$ or $zPxPy$. Applying Lemma 1 in Appendix A.1 we see that $\rho(x,\{x,y\}) - \rho(x,\{x,y,z\}) = 0$ and $\rho(z,\{y,z\}) - \rho(z,\{x,y,z\}) = 0$. Since both μ_1 and μ_2 rationalize the same choice function, we can conclude that the set $T_{\mathcal{P}_1} = T_{\mathcal{P}_2}$. Thus, there is no $P \in \mathcal{P}_{\mu_2}$ such that $xPzPy$ or $zPxPy$. We can prove the statement analogously for the reverse linear order over the triple, and thus, for all triples. Therefore, μ_2 is peak-monotone as well.

A.5. Proof of Corollary 4

The characterization result follows immediately from Apesteguia et al. (2017, Corollaries 1 and 2) and Theorem 1. In order to prove the relationship between choice probabilities from binary sets and larger sets for single-peaked RUMs, consider a stochastic choice function generated by a single-peaked on a line RUM. We know from Apesteguia et al. (2017, Corollary 1) that it satisfies REG and SCEN. It follows from Claim 2 in the proof of Theorem 1 in Apesteguia et al. (2017) that for all subsets $M = \{x_1,\dots,x_{|M|}\}$ and $x_1Lx_2L\dots Lx_{|M|}$ a stochastic choice function that satisfies REG and SCEN is such that

$$\rho(x_1,M) = \rho(x_1,\{x_1,x_2\}), \rho(x_{|M|},M) = \rho(x_{|M|},\{x_{|M|-1},x_{|M|}\}), \text{ and}$$
$$\rho(x_i,M) = \rho(x_i,\{x_{i-1},x_i,x_{i+1}\}).$$

We apply SCEN to obtain

$$\begin{aligned}\rho(x_i,\{x_{i-1},x_i,x_{i+1}\}) &= 1-\rho(x_{i-1},\{x_{i-1},x_i,x_{i+1}\}) - \rho(x_{i+1},\{x_{i-1},x_i,x_{i+1}\})\\ &= 1-\rho(x_{i-1},\{x_{i-1},x_i\}) - \rho(x_{i+1},\{x_i,x_{i+1}\})\end{aligned}$$

and the result follows.

In order to show how choice probabilities from binary sets can be used to obtain choice probabilities from larger sets when generated by a single-dipped RUM, recall that such stochastic choice functions satisfy REG and EXT (Apesteguia et al., 2017, Corollary 2). Thus, for all subsets $M = \{x_1, \ldots, x_{|M|}\}$ and $x_1 L x_2 L \ldots L x_{|M|}$ it holds that $\rho(x_i, \{x_{i-1}, x_i, x_{i+1}\}) = 0$ for all $i \in [2, |M| - 1]$. REG implies that $\rho(x_i, M) = 0$ for all $i \in [2, |M| - 1]$ as well. Therefore, $1 = \rho(x_1, M) + \rho(x_{|M|}, M)$ and REG ensures that the final result holds.

A.6. Proof of Proposition 1

Sufficiency. Consider a stochastic choice function ρ satisfying REG and CEN. Rationalizability is guaranteed by Apesteguia et al. (2017, Theorem 1). Fix RUM μ that rationalizes ρ and consider a triple $\{x, y, z\} \subseteq X$ such that $xLyLz$. We distinguish two cases: (a) $\rho(y, \{x, y, z\}) = 0$ and (b) $\rho(y, \{x, y, z\}) > 0$. If $\rho(y, \{x, y, z\}) = 0$ this means that $\nexists P' \in \mathcal{P}_\mu$ such that $y \in t(\{x, y, z\}, P')$, and the peak-pit on a line condition is trivially satisfied. In case (b), $\rho(y, \{x, y, z\}) > 0$ implies that $\exists P' \in \mathcal{P}_\mu$ such that $y \in t(\{x, y, z\}, P')$. We know from CEN and Lemma 1 in Appendix A.1 that $\rho(z, \{y, z\}) - \rho(z, \{x, y, z\}) = \sum_{P \in \mathcal{P} : xPzPy} \mu(P) = 0$. Therefore, $\nexists P \in \mathcal{P}_\mu$ such that $xPzPy$. Similarly, for the other extreme alternative x, we can infer from CEN and Lemma 1 in Appendix A.1 that $\nexists P \in \mathcal{P}_\mu$ such that $zPxPy$. Hence, the profile $\mathcal{P}_\mu$ is peak-pit on a line, as well as the supports of all other rationalizing RUMs.

Necessity. Consider a stochastic choice function ρ generated by a peak-pit on a line RUM μ. REG is a necessary condition for rationalizability, and is thus satisfied by ρ. Take a triple $xLyLz$ with $y \in t(\{x, y, z\}, P')$ for some $P' \in \mathcal{P}_\mu$. Therefore, $\rho(y, \{x, y, z\}) > 0$. The peak-pit on a line property implies $\nexists P \in \mathcal{P}_\mu$ with $xPzPy$ and $zPxPy$. Applying Lemma 1 in Appendix A.1 gives that $\rho(x, \{x, y\}) - \rho(x, \{x, y, z\}) = 0$ and $\rho(z, \{y, z\}) - \rho(z, \{x, y, z\}) = 0$, which coincides with the definition of CEN. If, on the other hand, there is no $P' \in \mathcal{P}_\mu$ with $y \in t(\{x, y, z\}, P')$, then y is never chosen from the triple and $\rho(y, \{x, y, z\}) = 0$, thus CEN is trivially satisfied.

A.7. Stochastic choices generated by a peak-pit on a line RUM

We first show how the complete stochastic choice function generated by a peak-pit on a line RUM can be recovered from binary choice probabilities.

Corollary 5. *Let ρ be a stochastic choice function generated by a peak-pit on a line RUM. Consider a subset of alternatives $M = \{x_1, \ldots, x_{|M|}\} \subseteq X$ with $|M| \geqslant 3$ and $x_1 L x_2 L \ldots L x_{|M|}$ and let $N = \{x_j \in M : \forall i \in [1, j-1]$ and $\forall k \in [j+1, |M|], \exists P \in \mathcal{P}_\mu$ such that $x_j \in t(\{x_i, x_j, x_k\}, P)\} \cup \{x_1, x_{|M|}\}$. It holds that*

$$\begin{aligned}
\rho(x_j, M) &= 0, \text{ if } x_j \notin N,\\
\rho(x_1, M) &= \rho(x_1, \{x_1, x_k\}), \text{ such that } x_k \in N \text{ and } \nexists x_{k'} \in N \text{ with } k' \in (1, k),\\
\rho(x_{|M|}, M) &= \rho(x_{|M|}, \{x_i, x_{|M|}\}), \text{ such that } x_i \in N \text{ and } \nexists x_{i'} \in N \text{ with } i' \in (i, |M|),\\
\rho(x_j, M) &= \rho(x_j, \{x_i, x_j\}) - \rho(x_k, \{x_j, x_k\}), \text{ if } \{x_i, x_j, x_k\} \in N \text{ with } i < j < k \text{ and}\\
&\qquad \nexists x_{i'}, x_{k'} \in N \text{ with } i' \in (i, j) \text{ and } k' \in (j, k).
\end{aligned}$$

Proof. Observe that all alternatives $x_j \notin N$ are positioned middle in some triple $\{x_i, x_j, x_k\} \in M$ and there is no preference that ranks x_j top in that triple. Hence, the profile is locally single-dipped for that triple and $\rho(x_j, \{x_i, x_j, x_k\}) = 0$. REG implies that $\rho(x_j, M) = 0$. Therefore, it holds that

$$\sum_{x_j \in N} \rho(x_j, N) = 1 = \sum_{x_j \in N} \rho(x_j, N) + \sum_{x_j \in M \setminus N} \rho(x_j, M) = \sum_{x_j \in M} \rho(x_j, M).$$

REG implies that $\rho(x_j, N) = \rho(x_j, M)$ for all $x_j \in N$. Note also that since the support is peak-pit over the set M, it has to be single-peaked over N since in each triple the middle alternative is ranked top for at least one preference in the profile. The choice probabilities $\rho(x_j, M)$ for $x_j \in N$ expressed in terms of binary choice probabilities follow directly from Corollary 4. □

In order to see how this result can be applied for choice prediction, consider a stochastic choice function that satisfies REG and CEN. It follows from Proposition 1 that preferences are revealed to be peak-pit. Suppose now that there is a perturbation of the state probabilities and consider a set M. Let $N = \{x_j \in M : \forall i \in [1, j-1]$ and $\forall k \in [j+1, |M|], \rho(x_j, \{x_i, x_j, x_k\}) > 0\} \cup \{x_1, x_{|M|}\}$. Since the support is unaffected by the shift in state probability distribution, the set of alternatives never chosen from a triple stay the same and the researcher can use the set N to recover the new stochastic choice function using binary choice probabilities as shown in Corollary 5.

A.8. Proof of Proposition 2

Sufficiency. Consider a rationalizable stochastic choice function ρ satisfying L-CEN. Fix RUM μ that rationalizes ρ and consider a triple $\{x, y, z\} \subseteq X$. We distinguish two possible cases: ρ over the triple satisfies (1) L-SCEN or (2) L-EXT. Consider the first case. Because of L-SCEN there exists at least one alternative, let it be y, for which $\rho(x, \{x, y\}) = \rho(x, \{x, y, z\})$ and $\rho(z, \{y, z\}) = \rho(z, \{x, y, z\})$.

Applying Lemma 1 from Appendix A.1 gives $\rho(x,\{x,y\})-\rho(x,\{x,y,z\})=\sum_{P\in\mathcal{P}:zPxPy}\mu(P)=0$ and $\rho(z,\{y,z\})-\rho(z,\{x,y,z\})=\sum_{P\in\mathcal{P}:xPzPy}\mu(P)=0$, i.e. there is no preference $P\in\mathcal{P}_\mu$ for which $zPxPy$ or $xPzPy$, meaning that $y\notin b(\{x,y,z\},P)$ for all preferences in the support, and the support satisfies the peak-pit condition over this triple.

Next, consider the second case. L-EXT implies that there is an alternative in the triple, say y, for which $\rho(y,\{x,y,z\})=0$. Therefore, there is no $P\in\mathcal{P}$ such that $yPxPz$ and $yPzPx$, which means that $y\notin t(\{x,y,z\},P)$ for all preferences in the support and the peak-pit condition holds over the triple.

Necessity. Consider a locally peak-pit RUM μ that generates a stochastic choice function ρ. Take a triple $\{x,y,z\}\subseteq X$. We distinguish again two possible cases: (1) $x\notin t(\{x,y,z\},P)$ and (2) $x\notin b(\{x,y,z\},P)$ for all $P\in\mathcal{P}$. Consider the first case. Since there is an alternative x in $\{x,y,z\}$ for which $x\notin t(\{x,y,z\},P)$ for all preferences in $\mathcal{P}$, $\sum_{P\in\mathcal{P}:xPyPz}\mu(P)+\sum_{P\in\mathcal{P}:xPzPy}\mu(P)=0$ and $\rho(x,\{x,y,z\})=0$ and L-CEN is satisfied.

Finally, let $x\notin b(\{x,y,z\},P)$ for all $P\in\mathcal{P}$. Therefore, $\sum_{P\in\mathcal{P}:yPzPx}\mu(P)=0$ and $\sum_{P\in\mathcal{P}:zPyPx}\mu(P)=0$. Using Lemma 1 in Appendix A.1 we see that ρ is such that $\rho(z,\{x,z\})=\rho(z,\{x,y,z\})$ and $\rho(y,\{x,y\})=\rho(y,\{x,y,z\})$ and hence L-SCEN holds, and thus L-CEN.

A.9. Proof of Proposition 3

Sufficiency. Consider a rationalizable stochastic choice function ρ satisfying L-CEN or L-ISA. Fix RUM μ that rationalizes ρ and consider a triple $\{x,y,z\}\subseteq X$. We distinguish three possible cases: ρ over the triple satisfies (1) L-SCEN, (2) L-EXT, or (3) L-ISA. We know from Proposition 2 that stochastic choice functions satisfying (1) or (2) are rationalizable by locally peak-pit RUMs, hence with triple-wise value-restricted RUMs.

Assume that the stochastic choice function satisfies L-ISA over the triple, and therefore $\rho(x,\{x,y\})=\rho(x,\{x,z\})=\rho(x,\{x,y,z\})$. Lemma 1 from Appendix A.1 implies $\sum_{P\in\mathcal{P}:zPxPy}\mu(P)=0$ and $\sum_{P\in\mathcal{P}:yPxPz}\mu(P)=0$. This means that all preferences P in $\mathcal{P}_\mu$ are such that $x\notin m(\{x,y,z\},P)$, and triple-wise value-restriction holds over the triple. Therefore, all rationalizing RUMs are triple-wise value-restricted.

Necessity. Consider a triple-wise value-restricted RUM μ that generates a stochastic choice function ρ. Take a triple $\{x,y,z\}\subseteq X$. We distinguish again three possible cases: (1) $x\notin t(\{x,y,z\},P)$, (2) $x\notin m(\{x,y,z\},P)$, and (3) $x\notin b(\{x,y,z\},P)$ for all $P\in\mathcal{P}$. Proposition 2 implies that the generated stochastic choice function satisfies L-CEN in cases (1) and (3).

Consider the case in which $x\notin m(\{x,y,z\},P)$ for all $P\in\mathcal{P}$. This means that $\sum_{P\in\mathcal{P}:yPxPz}\mu(P)=0$ and $\sum_{P\in\mathcal{P}:zPxPy}\mu(P)=0$. Lemma 1 in Appendix A.1 implies that $\rho(x,\{x,y\})=\rho(x,\{x,z\})=\rho(x,\{x,y,z\})$. Hence, L-ISA is satisfied.

A.10. Proof of Proposition 4

Consider a stochastic choice function ρ that satisfies L-CEN or L-ISA and take an arbitrary triple $\{x,y,z\}\subseteq X$. The choice function over $\{x,y,z\}$ satisfies therefore L-SCEN or L-EXT or L-ISA. Suppose first that the choice function satisfies L-SCEN and $\rho(x,\{x,y\})=\rho(x,\{x,y,z\})$ and $\rho(z,\{y,z\})=\rho(z,\{x,y,z\})$. REG implies $\rho(x,\{x,y,z\})\leqslant\rho(x,\{x,z\})$, hence $\rho(x,\{x,y\})\leqslant\rho(x,\{x,z\})$. Analogously, we obtain for the choice probabilities of alternative z that $\rho(z,\{y,z\})\leqslant\rho(z,\{x,z\})$. We can rewrite the last inequality as $1-\rho(y,\{y,z\})\leqslant 1-\rho(x,\{x,z\})$, therefore $\rho(x,\{x,z\})\leqslant\rho(y,\{y,z\})$. All inequalities are summarized below:

$$\rho(x,\{x,y\})\leqslant\rho(x,\{x,z\})\leqslant\rho(y,\{y,z\}). \tag{2}$$

Suppose next that the choice over the triple satisfies L-EXT such that $\rho(x,\{x,y,z\})=0$. Since $\rho(x,\{x,y,z\})+\rho(y,\{x,y,z\})+\rho(z,\{x,y,z\})=\rho(y,\{x,y,z\})+\rho(z,\{x,y,z\})=\rho(y,\{y,z\})+\rho(z,\{y,z\})=1$ it follows from REG that $\rho(y,\{x,y,z\})=\rho(y,\{y,z\})\leqslant\rho(y,\{x,y\})$ and $\rho(z,\{x,y,z\})=\rho(z,\{y,z\})\leqslant\rho(z,\{x,z\})$, which can be summarized as

$$\rho(x,\{x,z\})\leqslant\rho(y,\{y,z\})\leqslant\rho(y,\{x,y\}). \tag{3}$$

And finally if the choice function over the triple satisfies L-ISA it holds w.l.o.g. that

$$\rho(x,\{x,y\})=\rho(x,\{x,z\})=\rho(x,\{x,y,z\}). \tag{4}$$

Next, we check whether moderate stochastic transitivity holds for all possible cases:

- Case 1: $\rho(y,\{x,y\})\geqslant\frac{1}{2}$ and $\rho(x,\{x,z\})\geqslant\frac{1}{2}$. We need to show that $\rho(y,\{y,z\})\geqslant\min\{\rho(y,\{x,y\}),\rho(x,\{x,z\})\}$. If L-SCEN or L-EXT hold then Inequalities (2) and (3) imply that $\rho(y,\{y,z\})\geqslant\rho(x,\{x,z\})$. If L-ISA holds, it follows from (4) that $\frac{1}{2}\geqslant\rho(x,\{x,y\})=\rho(x,\{x,z\})\geqslant\frac{1}{2}$. Therefore, $\rho(y,\{x,y\})=\frac{1}{2}$, which contradicts the assumption $\max\{\rho(y,\{x,y\}),\rho(x,\{x,z\})\}>\frac{1}{2}$ and therefore, moderate transitivity is trivially satisfied.
- Case 2: $\rho(y,\{y,z\})\geqslant\frac{1}{2}$ and $\rho(z,\{x,z\})\geqslant\frac{1}{2}$. We will prove that $\rho(y,\{x,y\})\geqslant\min\{\rho(y,\{y,z\}),\rho(z,\{x,z\})\}$. Inequality (2) states that $\rho(x,\{x,y\})\leqslant\rho(x,\{x,z\})$, which can be rewritten as $\rho(y,\{x,y\})\geqslant\rho(z,\{x,z\})$. Similarly, it follows from (3) that $\rho(y,\{x,y\})\geqslant\rho(y,\{y,z\})$. Equality (4) implies that $\rho(y,\{x,y\})=\rho(z,\{x,z\})$.
- Case 3: $\rho(x,\{x,y\})\geqslant\frac{1}{2}$ and $\rho(y,\{y,z\})\geqslant\frac{1}{2}$. Moderate stochastic transitivity holds if $\rho(x,\{x,z\})\geqslant\min\{\rho(x,\{x,y\}),\rho(y,\{y,z\})\}$. Inequality (2) implies $\rho(x,\{x,z\})\geqslant\rho(x,\{x,y\})$. Since we have $\rho(y,\{x,y\})\leqslant\frac{1}{2}$, we obtain using (3) and $\rho(y,\{y,z\})\geqslant\frac{1}{2}$ that $\rho(x,\{x,y\})=\rho(y,\{y,z\})=\frac{1}{2}$ contradicting the assumption that at least one of the two probabilities is strictly larger than $\frac{1}{2}$. Hence

if L-EXT holds, moderate stochastic transitivity is trivially satisfied. If L-ISA holds, then from Equality (4) we have $\rho(x,\{x,z\}) = \rho(x,\{x,y\})$.

– Case 4: $\rho(x,\{x,z\}) \geqslant \frac{1}{2}$ and $\rho(z,\{y,z\}) \geqslant \frac{1}{2}$. We will show that $\rho(x,\{x,y\}) \geqslant \min\{\rho(x,\{x,z\}),\rho(z,\{y,z\})\}$. Since $\rho(y,\{y,z\}) \leqslant \frac{1}{2}$, Inequality (2) implies that $\rho(x,\{x,z\}) = \rho(y,\{y,z\}) = \frac{1}{2}$. This contradicts, however, the assumption that at least one of the two probabilities is strictly larger than 0.5 and moderate stochastic transitivity is trivially satisfied. The same result holds for the case of L-EXT. For L-ISA, we have that $\rho(x,\{x,y\}) = \rho(x,\{x,z\})$.

Since the alternatives are arbitrarily chosen, the remaining two cases are analogous for L-SCEN to cases 3 and 4, and for L-EXT and L-ISA to cases 1 and 2.

However, REG and L-CEN or L-ISA are not sufficient to guarantee strong stochastic transitivity. Consider the case with $\rho(x,\{x,y\}) \geqslant \frac{1}{2}$ and $\rho(y,\{y,z\}) \geqslant \frac{1}{2}$. We know from Inequality (2) that $\rho(x,\{x,z\}) \leqslant \rho(y,\{y,z\})$. Strong stochastic transitivity would be satisfied in this case if $\rho(x,\{x,z\}) = \rho(y,\{y,z\})$, which does not necessarily need to hold.

A.11. Proof of Proposition 5

We prove the result using the alternative definition of the peak-monotone domain provided in Lemma 2 in Appendix A.4. Let $T_\rho = \{x \in X : \rho(x,X) > 0\}$ be a set containing alternatives chosen with a positive probability from the set of alternatives X. Observe that if μ is a rationalizing RUM, then $T_\rho = T_{\mathcal{P}_\mu} = \bigcup_{P\in\mathcal{P}_\mu} t(X,P) = T$.

Sufficiency. Consider a stochastic choice function ρ satisfying WCEN. Fix RUM μ that rationalizes ρ and consider a triple $\{x,y,z\} \subseteq X$ such that w.l.o.g. $xLyLz$ and $\rho(x,\{x,y,z\}) > 0$ and $\rho(y,\{x,y,z\}) > 0$. Hence, $\{x,y\} \subseteq T$ and SCEN holds on the triple. Using our insight from Theorem 1, we know that there is no preference $P \in \mathcal{Q}$ with $xPzPy$ or $zPxPy$ for all such triples. Our result in Lemma 2 shows that peak-monotonicity is satisfied.

Necessity. Consider a peak-monotone RUM μ that generates a stochastic choice function ρ. Take a triple $\{x,y,z\}$ and assume w.l.o.g. that $xLyLz$. Suppose that $\{x,y\} \subseteq T$. Lemma 2 implies there is no $P \in \mathcal{Q}$ such that $xPzPy$ or $zPxPy$. Applying Lemma 1 in Appendix A.1 we see that $\rho(x,\{x,y\}) = \rho(x,\{x,y,z\})$ and $\rho(z,\{y,z\}) = \rho(z,\{x,y,z\})$ and WCEN holds.

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